Velocity Transformation in SR

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Without rigorous proof I present the following vector formula for addition of velocities in 3 dimensions. I derived (inferred?) it from the common velocity addition formula presented in introductory SR texts. I use this formula all the time for transforming velocities of moving objects between reference frames. It works for objects with velocity magnitude c too;

Given:

- An inertial reference frame S for which the velocity of an object is specified as u.
- inertial reference frame S' with velocity v relative to S.

The velocity of the object u' with respect to S' is given by.

$$\vec{u'} = \left(\frac{\vec{u_{\|v}} - \vec{v}}{\left(1 - \left(\frac{\vec{u}\cdot\vec{v}}{c^2}\right)\right)} \right) + \left(\frac{\vec{u_{\perp v}}}{\gamma\left(1 - \left(\frac{\vec{u}\cdot\vec{v}}{c^2}\right)\right)} \right)$$

Where:

- $\vec{u}$ is a 3 dimensional vector specifying the velocity of an object with respect to reference frame S.
- $\vec{u'}$ is a 3 dimensional vector specifying the velocity of the same object with respect to reference frame S'.
- $\vec{v}$ is a 3 dimensional vector specifying the velocity of S' with respect to S. (u and v do not have to be parallel.)
- $\vec{u_{\|v}}$ is that portion of velocity $\vec{u}$ which is parallel to $\vec{v}$.
- $\vec{u_{\perp v}}$ is that portion of velocity $\vec{u}$ which is perpendicular to $\vec{v}$.
- $\vec{u}\cdot\vec{v}$ is the scalar product of vectors $\vec{u}$ and $\vec{v}$ (dot product).
- γ is the relativistic gamma factor for the velocity of S' with respect to S.

Substituting for parallel and perpendicular vectors, I get.

$$\vec{u'} = \left(\frac{(\vec{u}\cdot\hat{v})\hat{v} - \vec{v}}{\left(1 - \left(\frac{\vec{u}\cdot\vec{v}}{c^2}\right)\right)} \right) + \left(\frac{\vec{u} - (\vec{u}\cdot\hat{v})\hat{v}}{\gamma\left(1 - \left(\frac{\vec{u}\cdot\vec{v}}{c^2}\right)\right)} \right)$$

Where: $\hat{v}$ is the unit vector of $\vec{v}$. (the direction of $\vec{v}$).